Chaos in closed isotropic cosmological models with steep scalar field potential

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Abstract

The dynamics of closed scalar field FRW cosmological models is studied for several types of exponentially and more than exponentially steep potentials. The parameters of scalar field potentials which allow a chaotic behaviour are found from numerical investigations. It is argued that analytical studies of equation of motion at the Euclidean boundary can provide an important information about the properties of chaotic dynamics. Several types of transition from chaotic to regular dynamics are described.

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1 Introduction

The studies of chaotical dynamics of closed isotropic cosmological model has a long story. They were initiated by papers [1] where the possibility to avoid a singularity at the contraction stage in such a model with a minimally coupled massive scalar field was discovered. Later it was found that this model allows the existence of periodical trajectories [2] and aperiodical infinitely bouncing trajectories having a fractal nature [3]. This result was reproduced in other terms in our papers [4, 5]. In [6] the set of periodical trajectories was studied from the viewpoint of dynamical chaos theory. It was proved that the dynamics of a closed universe with a massive scalar field is chaotic and an important invariant of the chaos, the topological entropy, was calculated. On the other hand, in our paper [5] we found that deformations of the scalar field potential may change the dynamical regime from chaotic to regular one. This fact poses a question: is the chaotic behaviour closely connected with a concrete form of the potential ($V(\varphi) = \frac{m^2\varphi^2}{2}$) used in the previous papers or is it a more general phenomenon?

A numerical analysis shows that the situation for another natural scalar field potential $V(\varphi) = \lambda\varphi^4$ is qualitatively the same. But most of modern scenarios based on ideas of the string theory and compactification naturally lead to another forms of potential which are exponential or behave as exponential for large φ (see, for example, Ref. [7]). This steepness of the potential apparently changes the possibilities of escaping the singularities and alters the structure of infinitely bouncing trajectories. Under some conditions, which will be studied below in detail, the chaotic behaviour can completely disappear. Another interesting problem concerning asymptotic regime near the singularity for such potentials was recently studied in [8].

The structure of the paper is the following: in Sec. 2 we recall the structure of chaos in the case of a massive scalar field with a special attention paid to the statements which are valid for an arbitrary scalar field potential. In Sec. 3 the conditions for the chaotic behaviour in models with exponential potentials are investigated both analytically and numerically. In Sec. 4 this analysis is extended to steeper potentials. Sec. 5 provides a brief summary of the result obtained.

2 Chaotic properties of closed FRW model with a scalar field

We shall consider a cosmological model with an action

$$S = \int d^4x\sqrt{-g}\left\{\frac{m_P^2}{16\pi}R + \frac{1}{2}g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - V(\varphi)\right\}. \tag{1}$$

For a closed Friedmann model with the metric

$$ds^2 = N^2(t)dt^2 - a^2(t)d^2\Omega^{(3)}, \tag{2}$$

where $a(t)$ is a cosmological radius, N – a lapse function and $d^2\Omega^{(3)}$ is the metric of a unit 3-sphere and with homogeneous scalar field φ the action (1) takes the form

$$S = 2\pi^2 \int dtNa^3 \left\{ \frac{3m_P^2}{8\pi} \left[-\left(\frac{\dot{a}}{Na} \right)^2 + \frac{1}{a^2} \right] + \frac{\dot{\varphi}^2}{2N^2} - V(\varphi) \right\}. \tag{3}$$

Now choosing the gauge $N = 1$ we can get the following equations of motion

$$\frac{m_P^2}{16\pi} \left(\ddot{a} + \frac{\dot{a}^2}{2a} + \frac{1}{2a} \right) + \frac{a\dot{\varphi}^2}{8} - \frac{aV(\varphi)}{4} = 0, \tag{4}$$

$$\ddot{\varphi} + \frac{3\dot{\varphi}\dot{a}}{a} + V'(\varphi) = 0. \tag{5}$$

In addition, we can write down the first integral of motion of our system

$$-\frac{3}{8\pi} m_P^2 (\dot{a}^2 + 1) + \frac{a^2}{2} \left(\dot{\varphi}^2 + 2V(\varphi) \right) = 0. \tag{6}$$

It is easy to see from Eq. (6) that the points of maximal expansion and those of minimal contraction, i.e. the points, where $\dot{a} = 0$ can exist only in the region where

$$a^2 \leq \frac{3}{8\pi} \frac{m_P^2}{V(\varphi)}, \tag{7}$$

Sometimes, the region defined by inequalities (7) is called Euclidean , and the opposite region is called Lorentzian. This definition is not good enough, because for this dynamical system there are no classically forbidden regions at all (see [11] for detail), but we shall use it for brevity. Now we would like to distinguish between the points of minimal contraction where $\dot{a} = 0$, $\ddot{a} > 0$ and those of

maximal expansion where $\dot{a} = 0$, $\ddot{a} < 0$. Assuming $\dot{a} = 0$, one can express $\dot{\varphi}^2$ from Eq. (6) as

$$\dot{\varphi}^2 = \frac{3}{4\pi}\frac{m_P^2}{a^2} - 2V(\varphi). \tag{8}$$

Substituting (8) and $\dot{a} = 0$ into (4) we have

$$\ddot{a} = \frac{8\pi V(\varphi)a}{m_P^2} - \frac{2}{a}. \tag{9}$$

From Eq. (9) one can easily see that the possible points of maximal expansion are localized inside the region

$$a^2 \leq \frac{1}{4\pi}\frac{m_P^2}{V(\varphi)} \tag{10}$$

while the possible points of minimal contraction lie outside this region (10) being at the same time inside the Euclidean region (7).

It is easy to see that the value of a on this separating curve is $\sqrt{2/3}$ times smaller than the corresponding a on the Euclidean boundary for a given value of the scalar field independently on the concrete form of $V(\varphi)$. In [11] it was shown that this ratio remains constant for some cases in a more general situation with a non-minimally coupled scalar field.

Here we would like to describe briefly the approach presented in [4]. The main idea consists in the fact that in the closed isotropical model with a minimally coupled scalar field satisfying the energodominance condition all the trajectories have the point of maximal expansion. Then the trajectories can be classified according to localization of their points of maximal expansion. The points of maximal expansion are all located inside the Euclidean region. A numerical investigation shows that this area has a quasi- periodical structure.

Narrow zones starting from which the trajectory has the point of bounce are separated by wide zones containing the initial conditions of trajectories falling into a singularity. Each zone of bounces contains a periodical trajectory with the point of full stop ($\dot{a} = 0$; $\dot{\varphi} = 0$) on the Euclidean boundary. Then studying the substructure of these zones from the point of view of possibility to have two bounces one can see that this substructure reproduce on the qualitative level the structure of the whole region of possible points of maximal expansion. Continuing this procedure *ad infinitum* yields the fractal set of infinitely bouncing trajectories. This kind of fractal, nonattracting invariant set is typical for chaotic systems without dissipation (see, for example, [9, 10]). In the theory of dynamical chaos it is called strange repellor.

Numerical investigations show also that the structure of periodical trajectories for the system Eqs (4)- (6) have two important properties:

1. All the simple periodical trajectories (i.e. having only one bounce per period) have a full stop point on the Euclidean boundary.

This property allows us to use the points of the Euclidean boundary and zero velocities as an initial conditions for searching a possible strange repellors.

2. Trajectories, going from the boundary into the euclidean region has a point of maximal expansion almost immediately, and then go towards a singularity. So, periodical trajectory approaches their bounce point on the boundary from the Lorentzian side.

The only exception is a single peculiar periodical trajectory existing in the case of nonzero cosmological constant term (see below).

This two properties were found in numerical analysis and the question whether or not they are satisfied for an arbitrary scalar field potential requires more investigation. Our researches show that it is true at least for potentials steep as power-law and steeper.

If the aforesaid is satisfied, some analytical approach is possible. Indeed, periodical trajectories with a full stop point on the Euclidean boundary penetrate into the Lorentzian region. That is why all such full stop points lie to the left from a critical point introduced by Page [3] in the configuration space (a, φ) . It is a point on the Euclidean boundary separating trajectories going into Lorentzian and Euclidean regions. By definition, the Page's point is the point where the direction of motion at the initial moment coincides with the direction of the tangent to the curve given by equality in (7) , i.e. the point where

$$\frac{\ddot{\varphi}}{\ddot{a}} = \frac{d\varphi}{da}. \tag{11}$$

Using Eqs. (4), (5),and (7) one can find for the case $V(\varphi) = \frac{m^2\varphi^2}{2}$

$$\begin{aligned} \varphi_{page} &= \sqrt{\frac{3}{4\pi}} m_P; \\ a_{page} &= 1/m, \end{aligned} \tag{12}$$

except for the trivial solution $\varphi = 0$, $a = \infty$.

Trajectories starting from $\varphi < \varphi_{page}$ and going into the Euclidean region reach the point of maximal expansion almost immediately after crossing the separating curve (see [4] where we studied such set of maximal expansion points). After that point the trajectory goes towards singularity. So it looks like a small

zigzag but not a "true" bounce. To define a really significant bounce, Cornish and Shellard in [6] used a condition that in the point of bounce $a < 1/m$ in addition to $\ddot{a} > 0$; $\dot{a} = 0$. This is just the condition that bounces lie to the left from the Page point in configuration space (a, φ).

It is also possible to use the criterion that the trajectory must return to the Lorentzian region after the bounce . Qualitatively the picture of regions containing bouncing trajectories under this criterion is the same as for the former one, with the only exception that the width of the regions is somewhat smaller. This criterion can be treated as more direct one, but it requires numerical integration while the study of Page's points may be done analytically. Indeed, it is easy to obtain an analog to (12) for an arbitrary potential during the same procedures. The result is the following equation for the φ-coordinate:

$$V(\varphi_{page}) = \sqrt{\frac{3m_P^2}{16\pi}} V'(\varphi_{page}). \tag{13}$$

It can be easily derived from Eq.(13) that a qualitative picture of massive scalar field case do not change for any pow-law potential with the even index: there is one non-trivial Page's point with the full stop points of periodical trajectories lying to the left from it on the euclidean boundary. In the next section we will see that the situation changes significantly for the exponentially steep potentials.

We finish this section by the descriptions of another important modification of the scalar field potential – introducing a constant term (so called cosmological constant). A non-zero value of the cosmological constant Λ makes possible an existence of trajectories without points of maximal expansion, i.e. the trajecto-

ries which begin and end in DeSitter regime

$$a(t) \sim \exp(-Ht), \quad t \to -\infty;$$

$$a(t) \sim \exp(Ht), \quad t \to \infty \tag{14}$$

or the trajectories which begin in the singularity and end in DeSitter expansion or vice versa. Here, in Eq. (14) H denotes Hubble constant

$$H = \sqrt{\frac{\Lambda}{3}}.$$

The chaos in the model with a massive scalar field and a cosmological constant

$$V(\varphi) = \frac{m^2\varphi^2}{2} + \frac{m_P^2}{8\pi}\Lambda \tag{15}$$

was studied in [5]. Before recalling some results of this paper, let us look on Page's points for this potential. The equation (13) has the form

$$\varphi^2 - \sqrt{\frac{3m_P^2}{4\pi}}\varphi + \frac{\Lambda}{m^2}\frac{m_P^2}{4\pi} = 0. \tag{16}$$

It is easy to see from (16) that for $\Lambda/m^2 > 0.75$ there are no such points: all trajectories from the Euclidean boundary go into the Lorentzian region (and then fall into DeSitter regime). For $\Lambda/m^2 < 0.75$ there are two Page's points. The left point has the same properties as for the potential without Λ, while properties of the right point are inverse: the trajectories with an initial $a > a_{page2}$ go into the Lorentzian region while trajectories with $a < a_{page2}$ (but, of course,$a > a_{page1}$) – to the Euclidean one. A numerical analysis confirms that all trajectories with $a > a_{page2}$ fall to DeSitter regime and can not belong to the chaotic repellor. So, the role of the 2-nd Page's point appear to be in restricting

of the chaotic area in the configuration space while the existence of chaos seems to be connected with the 1-st one.

An unstable periodical trajectory, found in [5] via numerical studies that restricts the area of bounce intervals in a from the large values of the scale factor lies to the left from the 2-nd Page's point. It crosses the φ-axis at some point with a scale factor a_m.

In addition to discussion in [5], we give here a simple analytical estimation of the location of this periodical trajectory in the configuration space. The typical value of a_m, corresponding to this trajectory can be easily estimated for large a. In this case we may assume the harmonic oscillations to be a solution for the field φ dynamics:

$$\varphi = A\cos(mt).$$

For calculating A consider a point where $\varphi = 0; \dot{a} = 0$. Taking the derivative with respect to time and using the constraint equation (6), we get

$$A^2 = \frac{m_P^2}{4\pi}\frac{3/a^2 - \Lambda}{m^2}.$$

Substituting these two expressions into (4) and integrating over one period T of the scalar field oscillation, we can find the velocity $\dot{a}$ after one oscillation:

$$\dot{a} = T\frac{1}{2}(\Lambda a - \frac{1}{a}).$$

It vanishes for

$$a_m = \sqrt{1/\Lambda}. \tag{17}$$

Trajectories with $a^2 > 1/\Lambda$ have increasing scale factors during the scalar field oscillations. They inevitably reach the DeSitter regime and can not belong to the strange repellor.

As we found in [5], if the cosmological constant is more than about one half of the mass square of the scalar field (a more accurate value is $\Lambda >\sim 0.28m^2$) the chaotic structure disappear and the dynamics becomes regular. This critical ratio Λ/m^2 is about 2.7 times lower than that corresponding to the disappearance of Page's points ($\Lambda/m^2 = 0.75$).

Surprisingly, the estimation (17) is correct even for Λ near the critical value when corresponding a_m can not be treated as large. Numerical values of a_m differ from (17) by several percents at most.

The possible values of Λ allowing the chaotic dynamics for potential (15) are also restricted from below for a negative cosmological constant. The corresponding critical value is $|\Lambda|/m^2 \sim 0.34$ (see [12] for detail).

3 Chaotic motion in exponential potentials

We start with one note about the pure exponential potential $V(\varphi) = M_0^4 \exp(\varphi/\varphi_0)$. It is easy to see from Eq.(13) that there are no Page's point in this case. The direction of a trajectory starting with zero initial conditions from the Euclidean boundary is fully determined by the value φ_0. If $\varphi_0 > \frac{\sqrt{3}}{4\sqrt{\pi}} m_P$ *all the trajectories* from the Euclidean boundary go into the Lorentzian region while if $\varphi_0 < \frac{\sqrt{3}}{4\sqrt{\pi}} m_P$ all of them go into the Euclidean region.

Now we consider the potential $V(\varphi) = M_0^4(\cosh(\varphi/\varphi_0) - 1)$. In the limit

$\varphi \to 0$ it looks like $\frac{m_{eff}^2 \varphi^2}{2}$ with $m_{eff} = \frac{M_0^2}{\varphi_0}$, while for large φ it looks like the pure exponential one. The equation for Page's point is now

$$\sinh \frac{\varphi}{\varphi_0} = \frac{\varphi_0}{\sqrt{3}} \frac{4\sqrt{\pi}}{m_P} (\cosh \frac{\varphi}{\varphi_0} - 1). \tag{18}$$

We can see that Page's point exists only if $\varphi_0 > \frac{\sqrt{3}}{4\sqrt{\pi}} m_P$. In the opposite case, again all the trajectories from the Euclidean boundary go into the Euclidean region. It means that there are no periodical trajectories with the full stop points in this case. Remembering the first property of our chaotic system listed in previous section, we may expect that the strange repellor is absent at all when $\varphi_0 < \frac{\sqrt{3}}{4\sqrt{\pi}} m_P$.

Results of the numerical integration confirms this analytical considerations. In Fig.1 several typical trajectories are plotted. Although points with $\dot{a} = 0$; $\ddot{a} > 0$ are possible, trajectories containing such points have a zigzag-like form and can not return to the Lorentzian region. Periodical trajectories are absent and the dynamics, in contrast to previously considered cases, is regular: it is impossible to avoid a singularity even on zero-measure set of initial condition. All the trajectories fall into singularity.

For $\varphi_0 > \frac{\sqrt{3}}{4\sqrt{\pi}} m_P$ the structure of trajectories is similar to the massive scalar field case. Again, if we consider points of maximal expansion, we can find regions corresponding to the bouncing trajectories. To distinguish between bounces and zigzags, we use an additional condition that the value of φ at the point of bounce is greater than φ_{page} (or we can use the condition that the trajectory after the bounce returns to the Lorentzian region). Then in limit $\varphi_0 \to \frac{\sqrt{3}}{4\sqrt{\pi}} m_P$ the width of bouncing regions tends to zero. It should be noted that when φ_0 crosses the

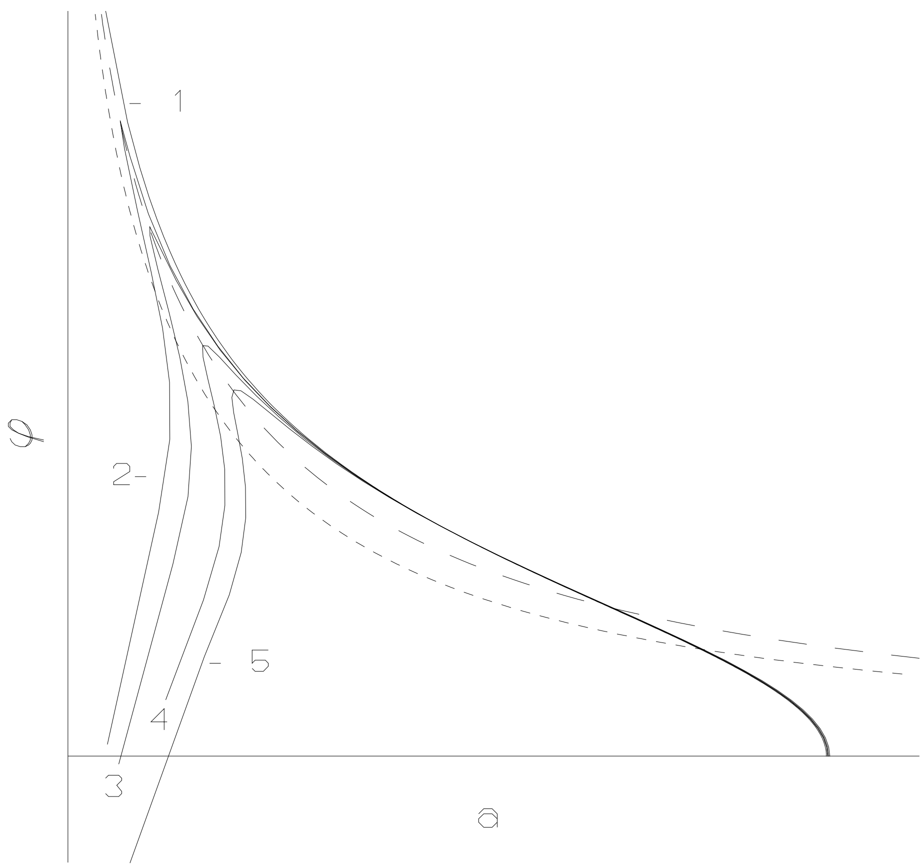

Figure 1: Example of trajectories with the initial conditions close to the boundary separating trajectories falling into $\varphi = +\infty$ (trajectory 1) and $\varphi = -\infty$ (trajectories $2-5$) singularities for the case $\varphi_0 < \frac{\sqrt{3}}{4\sqrt{\pi}} m_P$. This boundary is sharp, no fractal structure is present. Trajectories $2-5$ have a zigzag-like form, no periodical trajectories are present. The long-dashed line is the Euclidean boundary, the short-dashed line is the separating curve.

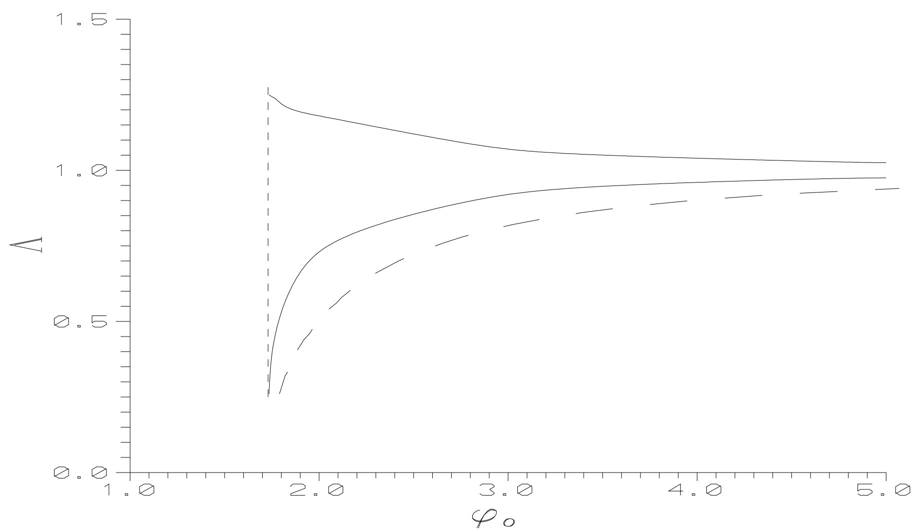

Figure 2: Several regions in the parameter plane for two-parameter family (19) which relate to a possibility of the chaotic dynamics. The parameter φ_0 is measured in units $m_P/\sqrt{16\pi}$. For $\varphi_0 < \sqrt{3}$ (to the left from the short-dashed curve) the dynamics is regular. For $\varphi_0 > \sqrt{3}$ the chaotic dynamics exists for Λ lying between two solid lines. The Page's points are absent below the long-dashed curve.

critical value the entire structure of periodical trajectories disappears in a jump-like manner.

Now we turn to a more general case $V(\varphi) = M_0^4(\cosh(\varphi/\varphi_0) - \Lambda)$. It corresponds to an effective cosmological constant $\Lambda_{eff} = \frac{8\pi M_0^4}{m_P^2}(1-\Lambda)$. Let us study the case of positive Λ_{eff}. Page's points exist if the transcendent equation

$$\sinh\frac{\varphi}{\varphi_0} = \frac{\varphi_0}{\sqrt{3}}\frac{4\sqrt{\pi}}{m_P}(\cosh\frac{\varphi}{\varphi_0} - \Lambda) \tag{19}$$

have roots.

If $\varphi_0 < \frac{\sqrt{3}}{4\sqrt{\pi}} m_P$ this equation has one root. The Page's point corresponding to this sole root looks like the 2-nd one for a massive scalar field with a cosmological constant: trajectories with $a > a_{page}$ tend to the Lorentzian region while

with $a < a_{page}$ – to the Euclidean one. And the dynamical picture obtained from the numerical integration have similar features: one unstable periodical orbit near $a^2 = 1/\Lambda_{eff}$ and nothing else. The discrepancy between the estimation (17) and the numerical solution grows with φ_0 reaching it's critical value.

The case $\varphi_0 > \frac{\sqrt{3}}{4\sqrt{\pi}} m_P$ is fully similar to the massive one with a cosmological constant: there exist two or zero Page's points depending on the values of φ_0 and Λ (the long-dashed curve in Fig.2). The chaos disappears for Λ (the lower solid curve in the same plot) which are somewhat larger than needed for Page's points to disappear.

For completeness we also present the area of values of $\Lambda_{eff} < 0$ leading to the chaotic dynamics (it is bounded by the upper solid curve in Fig.2). It essentially depends on the trajectory behaviour for large values of the scale factor and hence can not be understood by studying Page's points. We see from this plot that the massive case relation $|\Lambda_{eff}|/m_{eff}^2 \sim 0.34$ is still valid with a good accuracy.

4 Very steep potentials

For potentials more steep than exponential we face a situation which differs from the massive scalar field case significantly. It can be shown from studying the analog of (11) that trajectories from the Euclidean boundary for large values of φ go into Euclidean region. So the area of possible periodical trajectories is limited from the side of large φ. This leads to limiting the possible values of the scale factor on such trajectories. In Fig.3 we schematically show the number of

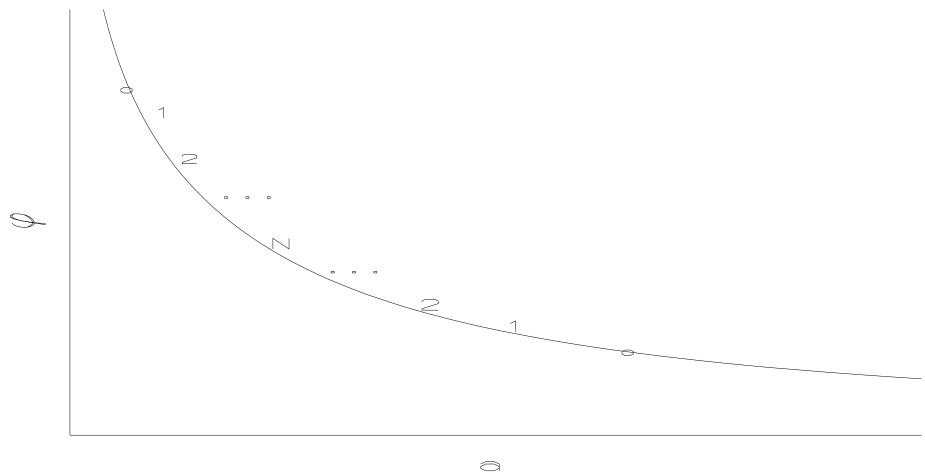

Figure 3: The relation between the number of the scalar field oscillations for trajectories starting from the Euclidean boundary and the position of their initial point in the case of very steep scalar field potentials.

oscillations of the field φ for periodical trajectories depending on the location of their full stop points on the Euclidean boundary. In contrast to the previous cases, this number is restricted from above by some number N. The numerical results show that the number of bounce intervals is also finite and restricted by some number $M > N$. In addition, there exists a complicated system of rules determining the substructure of intervals because the number of subintervals is now depends on the ordinal number of the interval. In particular, for an interval with ordinal number $N_1 > N$, the number of subintervals is always less than N. It is interesting that this dynamical picture looks very similar to that described in [12] for the system with a massive scalar field and a hydrodynamical matter.

Numerical investigations also show the transitions to regular dynamics which can be understood by studying the Page points configuration. Let us illustrate

this picture using some concrete one-parameter family of potentials

$$V(\varphi) = M_0^4(\exp(\varphi^2/\varphi_0^2) + \exp(-\varphi^2/\varphi_0^2) - 2).$$

Depending on the parameter φ_0, two or zero nontrivial Page,s points can appear. The critical value of φ_0 obtained numerically is $\varphi_0 \sim 0.905 m_P$. For φ larger than this value we have two Page's points. The area of initial points for trajectories penetrating into the Lorentzian region (i.e. possible full stop points of periodical trajectories) lies between them. So the properties of the Page's points are inverse with respect to those for the case of a massive scalar field with a cosmological constant.

In reality, the chaotic behaviour disappears for a somewhat larger value $\varphi \sim 0.96 m_P$. It is however remarkable that this quite simple analytical expressions (with computer used only to solve corresponding transcendent equation) gives us a rather good estimation of values of φ_0 which allows a chaotic regime.

The concrete form of potential for large φ does not affect the properties of chaos and therefore the steepness of the potential can be arbitrary high. The picture described above is valid even for cases with infinitely high potential walls. We have investigated a simple case

$$V = A/(\varphi_0^2 - \varphi^2) - A/\varphi_0^2.$$

In this case the condition of the existence of Page's points has the exact solution $\varphi_0 > \frac{9}{4\sqrt{\pi}} m_P$, while the numerical result for the existence of chaotic dynamics is $\varphi_0 > \frac{9.6}{4\sqrt{\pi}} m_P$.

5 Conclusions

We have studied the dynamics of closed Friedmann-Robertson-Walker universes with a scalar field. It was found that there exists a rather wide class of scalar field potentials, more steep than power-law, for which the dynamics of a scale factor is regular and there is no possibility to escape a singularity at a contraction stage.

On the other side, the class of potentials which allow the chaos, associated with chaotic oscillations of the scale factor, is also sufficiently wide. This chaos manifests itself in the presence of a strange repellor - a fractal set of unstable periodical orbits. Their existence is connected with a possibility for this system to have a bounce. Taking the point of maximal expansion as the initial one, the regions of the initial conditions in configuration space that lead to bouncing trajectories have a rather regular and obvious structure: the N-th region contains trajectories which have a bounce after N oscillations of field φ. More investigations is still required to study the possibility of existence of more complicated chaos for some another class of scalar field potentials.

The formal definition of the bounce ($\dot{a} = 0$, $\ddot{a} > 0$) is not good enough. The reason is that trajectories like in Fig.1 is very natural for our dynamical system. They satisfy the formal bounce criterion, but it is intuitively clear that zigzags on trajectories shown in Fig.1 can not change significantly the fate of the trajectories falling into singularity. On the other hand, the condition we used previously [4] (the value of scale factor at the next point of maximal expansion is greater than at previous one) rejects trajectories with decreasing but lying inside

the area of the strange repellor value of a at the points of maximal expansion. The condition that the trajectory after the bounce returns to the Lorentzian region or that the scale factor a at the point of bounce is less than the scale factor of the chaos generating Page's point may be used as a more suitable additional criterion.

Varying the potential, several types of transitions from chaotic to regular behaviour can be distinguished: (1) The number of bounce regions in the initial condition space can remains infinite, but the area of the strange repellor in the configuration space tends to zero (for the potential like $V(\varphi) = \frac{m^2\varphi^2}{2} + \Lambda$; $\Lambda > 0$). (2) The number of the bounce regions becomes finite and tends to zero while the area of the strange repellor remains unbounded (the same potential but for $\Lambda < 0$). (3) The number of the bounce regions diminishes and the strange repellor shrinks (for very steep potentials or for a mass-like potential in a more general case of a scalar field with a hydrodynamical matter [12]) (4) The width of the bounce regions tends to zero while their number and the area of the strange repellor remain unchanged (for $V(\varphi) = M_0^4(\cosh(\varphi/\varphi_0) - 1)$). The study of the transition from chaos to order for such dynamical systems may be interesting from the mathematical point of view.

Using the fact that the region in the configuration space where bounces are possible represents a sufficiently narrow area near the Euclidean boundary, we have proposed that studying trajectories having full stop points at this curve can provide us with an important information about the strange repellor as a whole. This suggestion realizes at least for the type of chaos which satisfies

two properties, established by numerical investigations for steep potentials and described in Sec.2. Their validity for less steep potentials will be the goal of our future work. In addition, the transition from chaos to order appear to be closely connected with the change of structure of Page's point configuration at the Euclidean boundary at least for positive scalar field potentials.

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